ON A PROBLEM FROM THE ELECTION MATHEMATICS

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Abstract.

The aim of this paper is to study a mathematical problem, related to a general voting scenario, in which a group of voters elects a committee from a pool of possible candidates. Every voter can vote positively or negatively for every candidate. The problem is to calculate the minimum number of positive votes, that each voter must make in order to elect at least a certain predefined lower bound of committee members. The authors have studied this problem in a previous paper, in which the committee members are elected by simple majority. The present paper generalises the result to other types of majorities, such as qualified majorities or more general parameter-based majorities. In the case when the group of voters is larger than the pool of candidates, a very good approximation of the answer is obtained, which does not depend on the total number of voters. The authors also analyze the error of this approximation, which happens to be at most one vote in absolute value.

Keywords: *election mathematics, simple and classified majority voting, pigeonhole principle*

Introduction

The research in this paper is based on the following problem. The general assembly of a university consists of 135 members. The assembly must fill in 24 vacant places in the academic council by an election procedure. There are 80 candidates for the places. Every member of the general assembly may vote for or against every candidate. A candidate is successfully chosen if he or she receives a simple majority of votes (that is at least 68 votes) in the election. We posed the following question:

How many positive votes must each member of the general assembly make in such a way that all vacant places in the academic council are surely filled in after one election procedure?

Intuitively, it is clear that such a number of positive votes exists. The authors have already obtained in [1] the answer to the question for simple majority voting. The present paper is concerned with exact and approximate estimations of the number from the question in other parameter-based voting scenarios.

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An exact general formula

We consider a general election scenario in which a group of **voters** elects a commission, consisting of certain number of **members**, from a pool of possible **candidates**. Every voter must give a positive or a negative vote for every candidate.

Let N be the number of voters. Let the election sheet contain m candidates. In this section a candidate is chosen for a member if he or she receives at least p positive votes, where $p \leq N$.

Let k be a positive integer, such that $k \leq m$ and let us denote

$$I = \frac{m.(p-1)+(k-1).(N-p+1)+1}{N} .$$

Theorem. *A sufficient condition for the successful election of at least k members of the commission is that every voter gives a positive vote for at least I^* candidates, where I^* is the least positive integer, such that $I^* \geq I$.*

Proof. We assume the opposite. This means that every voter has given at least I^* positive votes, but the number of successfully chosen members is l , where $l \leq k-1$.

We partition the positive votes of all voters in two non-intersecting classes: the first class contains the positive votes for candidates, which are chosen for members and the second class contains the positive votes for all the other non-chosen candidates.

The number of the chosen members is l , therefore the first class contains at most $l.N$ positive votes. The number of the non-chosen candidates is $m-l$. Each one of them has received at most $p-1$ positive votes. Therefore, the second class contains at most $(m-l).(p-1)$ positive votes.

So the total number of positive votes does not exceed

$$l.N+(m-l).(p-1) = m.(p-1)+l.(N-p+1)$$
$$\leq m.(p-1)+(k-1).(N-p+1) = I.N-1,$$

since $l \leq k-1$ and $N-p+1 \geq 0$. On the other hand, every voter has given at least I^* positive votes, so the total number of positive votes is at least $I^*.N \geq I.N$. We reached a contradiction. Therefore at least k members has been successfully chosen. *Q.E.D.*

From the proof it is clear that the number I^* is the least possible for the given k . It is easy to construct a particular voting, in which $k-1$ members are chosen with full majority (that is with N positive votes), each of the remaining non-chosen candidates receives at most $p-1$

positive votes and every voter gives exactly I^*-1 positive votes.

The proof of the theorem is closely connected with the pigeonhole principle. There is a threshold for the number of positive votes, such that exactly one more positive vote guarantees successful election.

In the example from the introduction we have $N=135,\ k=24,\ m=80,\ p=68$. Therefore, $I=\frac{1385}{27}\approx 51.2963$ and it is sufficient that every voter gives at least $I^*=52$ positive votes.

An approximate general formula, which does not depend on the total number of voters

From now on we assume the inequality $m\le N$. This is a natural assumption and it is true whenever the pool of candidates is part of the group of voters.

It is often the case that the number p of needed positive votes for successful election is a given portion of the number N of all possible votes.

For the purpose of this section, let α be a real number, such that $\alpha\in(0,1)$. We assume that p is the least positive integer, such that $p>\alpha.N$.

For example, $\alpha=\frac{1}{2}$ for a simple majority voting and $\alpha=\frac{2}{3}$ for a classified majority voting.

Let us denote $A=m.\alpha+(k-1).(1-\alpha)$. Thus A depends on k,m,α, but does not depend on N.

Theorem. *Let A^* be the least positive integer, such that $A^*\ge A$. Then $|A^*-I^*|\le 1$.*

Proof. We have $p-1\le\alpha.N<p$ by the minimality of p, which gives $\alpha.N<p\le\alpha.N+1$. Using these two inequalities we obtain

$$I\ge\frac{m.(\alpha.N-1)+(k-1).(N-\alpha.N-1+1)+1}{N}=A+\frac{1-m}{N}\ .$$

Therefore $A-I\le\frac{m-1}{N}\le 1$. We also obtain

$$I\le\frac{m.\alpha.N+(k-1).(N-\alpha.N+1)+1}{N}=A+\frac{k}{N}\ ,$$

hence $I-A\le\frac{k}{N}\le 1$. We have used that $k\le m\le N$. Finally,

$$A - I \le 1 \Rightarrow A \le I + 1 \le I^* + 1 \Rightarrow A^* \le I^* + 1$$

and similarly $I - A \le 1$ implies $I^* \le A^* + 1$. Therefore, $|A^* - I^*| \le 1$. *Q.E.D.*

As a corollary we obtain that A^*+1 positive votes from every voter are sufficient, but this is an upper bound, which sometimes is greater than the exact solution I^*. The last two sections give the precise relation between I^* and A^* for the cases of simple and classified majority votings.

Error estimation for simple majority voting

In this section we assume $\alpha = \frac{1}{2}$ and therefore

$$A = \frac{m+k-1}{2} \ .$$

Results from [1] easily imply that $I^* = A^* + 1$ in case N is even and $m-k$ is odd and $I^* = A^*$ in all other cases. For simple majority it never happens that $I^* = A^* - 1$.

In the example from the introduction $A = 51.5$ and N is odd, therefore $I^* = A^* = 52$.

Error estimation for classified majority voting

In this section we assume $\alpha = \frac{2}{3}$ and therefore

$$A = \frac{2m+k-1}{3} \ .$$

We note that A is an integer if and only if $m-k \equiv 2 \pmod 3$.

In the case $m-k \equiv 0 \pmod 3$ we have that $A + \frac{1}{3}$ is an integer, hence $A^* = A + \frac{1}{3}$.

In the case $m-k \equiv 1 \pmod 3$ we have that $A + \frac{2}{3}$ is an integer, hence $A^* = A + \frac{2}{3}$.

We consider three cases for the remainder of N modulo 3.

***Case* 1.** $N \equiv 0 \pmod 3$. Let $N = 3.x$, $x \ge 1$. Then $p = 2x+1$. We have

$$I = \frac{m.2x + (k-1).x + 1}{3.x} = A + \frac{1}{3.x} \ .$$

***Case* 1.1.** A is an integer ($m-k \equiv 2 \pmod 3$).

We obtain $A^*=A$ and $I^*=A^*+1$, since $A^*<I\le A+\frac{1}{3}<A^*+1$.

***Case* 1.2.** A is not an integer. Then $A^*=A+\frac{1}{3}$ or $A^*=A+\frac{2}{3}$. But $A<I\le A+\frac{1}{3}$ and therefore $A^*-1<A<I\le A^*$. Thus $I^*=A^*$ in this case.

***Case* 2.** $N\equiv 1\,(\mathrm{mod}\,3)$. Let $N=3.x+1$, $x\ge 0$. Then $p=2x+1$. We have

$$I=\frac{m.2x+(k-1).(x+1)+1}{3.x+1} \text{ and } A-I=\frac{2.(m-k)-1}{3.N}.$$

***Case* 2.1.** $k=m$. Then $I=A+\frac{1}{3.N}$. Of course, A is not an integer and by the same argument as in Case 1.2. we obtain $I^*=A^*$.

***Case* 2.2.** $k<m$. Since $1\le m-k<m\le N$, we obtain $\frac{1}{3.N}\le A-I\le\frac{2.N-1}{3.N}<\frac{2}{3}$.

***Case* 2.2.1.** A is an integer ($m-k\equiv 2\,(\mathrm{mod}\,3)$). Then $A^*=A$ and since $A^*-1<A-\frac{2}{3}<I<A=A^*$, we obtain $I^*=A^*$.

***Case* 2.2.2.** $m-k\equiv 0\,(\mathrm{mod}\,3)$. This means that $A^*=A+\frac{1}{3}$ and we have

$A^*-1=A-\frac{2}{3}<I<A<A^*$. Therefore, $I^*=A^*$.

Case 2.2.3. $m-k\equiv 1\,(\mathrm{mod}\,3)$. This means that $A^*=A+\frac{2}{3}$.

Case 2.2.3.1. $m-k<\frac{N+1}{2}$. Then $A-I<\frac{1}{3}$. We have

$A^*-1=A-\frac{1}{3}<I<A<A^*$. Thus $I^*=A^*$.

Case 2.2.3.2. $m-k\ge\frac{N+1}{2}$. Then $\frac{1}{3}\le A-I<\frac{2}{3}$. We obtain

$A^*-2=A-\frac{4}{3}<I\le A-\frac{1}{3}=A^*-1$. Therefore $I^*=A^*-1$.

Case 3. $N\equiv 2\,(\mathrm{mod}\,3)$. Let $N=3.x+2$, $x\ge 0$. Then $p=2x+2$. We have

$$I=\frac{m.(2x+1)+(k-1).(x+1)+1}{3.x+2} \text{ and } A-I=\frac{m-k-2}{3.N}.$$

Case 3.1. $k=m$. Then $I=A+\frac{2}{3.N}$. From $N\geq 2$ it follows $A<I\leq A+\frac{1}{3}$.

Since A is not an integer, $I^*=A^*$ as in Case 1.2.

Case 3.2. $k+1=m$. Then $I=A+\frac{1}{3.N}$. Again A is not an integer and $I^*=A^*$ as in the previous Case 3.1.

Case 3.3. $k+2=m$. Then $I=A$ and obviously $I^*=A^*$.

Case 3.4. $k+2<m$. Since $1\leq m-k-2<m\leq N$, we obtain $\frac{1}{3.N}\leq A-I<\frac{1}{3}$.

Case 3.4.1. A is an integer. Then $A^*=A$ and since $A^*-1<A-\frac{1}{3}<I<A=A^*$, we obtain $I^*=A^*$.

Case 3.4.2. A is not an integer. Then $A^*=A+\frac{1}{3}$ or $A^*=A+\frac{2}{3}$. But $A-\frac{1}{3}<I<A$, therefore $A^*-1\leq A-\frac{1}{3}<I<A<A^*$. Thus $I^*=A^*$.

More compactly, the analysis shows that:

when $N\equiv 0\ (\mathrm{mod}\ 3)$ and $m-k\equiv 2\ (\mathrm{mod}\ 3)$, we have $I^*=A^*+1$,

when $N\equiv 1\ (\mathrm{mod}\ 3)$, $m-k\equiv 1\ (\mathrm{mod}\ 3)$ and $m-k\geq\frac{N+1}{2}$, we have $I^*=A^*-1$,

in all other cases we have $I^*=A^*$.

Conclusion

A similar analysis for the relation between I^* and A^* can be done for many other rational values of α. What is more important, the formulated problem is general enough, so that it can be considered with more complicated voting systems. In a future research, the authors plan to apply the ideas from this paper, particularly those connected with the pigeonhole principle, for the voting scenarios, described in [2,3] and other extensive sources on election mathematics.

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